Reading questions.

1. Explain, in words what the material derivative represents.

The material derivative describes the rate of change seen by a moving "particle" of fluid

2. Why do we use a tensor to represent the state of stress of a material?

For a cube of material, the material's response to stress depends not only on the direction the stress is applied but the face. An "x" component of force applied to the face with a normal vector in the x direction will stretch the material, while the same force applied to another face would shear the material. We need 9 numbers to describe the state of stress.

3. What is the physical meaning of the divergence of the stress tensor?

It is the net force per unit volume acting at a point due to the internal state of stress of the material.

4. What is the differential form of conservation of mass for an incompressible flow?

Divergence of the velocity field is zero; $\nabla \cdot \vec{v}$

5. When is the assumption of incompressible flow a good one?

When the flow speed is quite a bit less than the speed of sound in the fluid.

6. What role does gravity play in a constant density flow?

It depends. If I have a closed container of constant density of the fluid, gravity cannot drive flow and gravity has no impact on the velocity field. Gravity does change the pressure that one would measure, but this hydrostatic component of pressure does not cause fluid motion. If the fluid is not contained, gravity can drive the fluid motion. For example, pour water on a sloped table and the water will flow downhill. It takes a little physical insight to decide if we can neglect or need to account for gravity.

7. What is the relationship between the stress tensor at a point and the stress vector acting on a surface passing through that point?

If I take the dot product of the surface normal vector with the stress tensor (think vector matrix multiply) then I get the stress vector acting at that point.

8. What's the difference between Euler's equations and the Navier-Stokes equations?

Viscosity is not present in Euler's formulation.

9. Physically, what is the reason that the stress tensor must be symmetric?

Conservation of angular momentum.

10. Physically, conceptually, and briefly, describe in words what is the constitutive law for a Newtonian fluid.

Stress is proportional to the rate of strain. The rate of strain is given by the velocity gradients.

11. What is the boundary condition on the fluid velocity at a solid wall (state in words or mathematically)?

Velocity is zero. Both normal and tangent to the surface.

12. What is vorticity and what does it represent?

Vorticity is computed from the curl of the velocity field. It represents the solid body rotation of a fluid particle.

13. What are the units of the stress tensor?

Stress, or force per unit area.

Estimates: (all numbers given in SI units)

1. A car traveling at 55 mph.

$\rho = 1.2, U = 25, L \approx 1, \mu = 1.8 \times 10^{-5}; \quad Re \approx 1.6 \times 10^{6}$.

2. A hurricane (with 80 mph winds over 100 miles)

$\rho = 1.2, U = 35, L \approx 160{,}000, \mu = 1.8 \times 10^{-5}; \quad Re \approx 3 \times 10^{10}$.

3. A bacteria in water (2 μm in size moving at 100 μm/s)

$\rho = 1000, U = 1 \times 10^{-4}, L = 2 \times 10^{-6}, \mu = 1 \times 10^{-3}; \quad Re = 2 \times 10^{-4}$.

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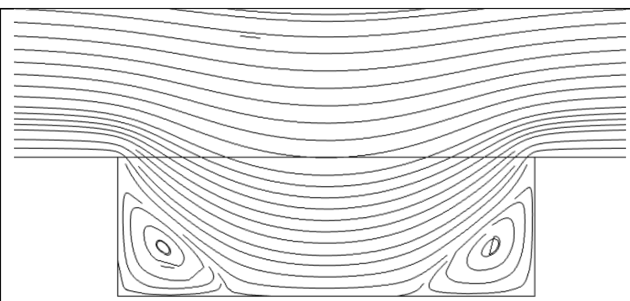
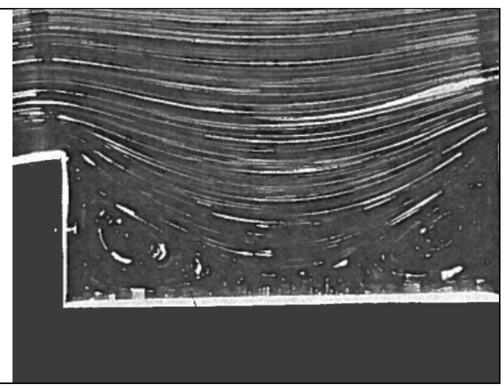
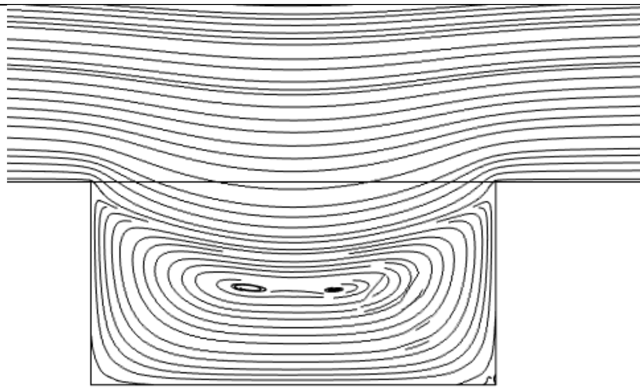
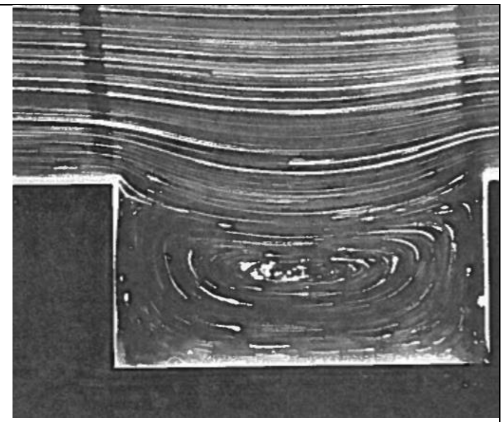
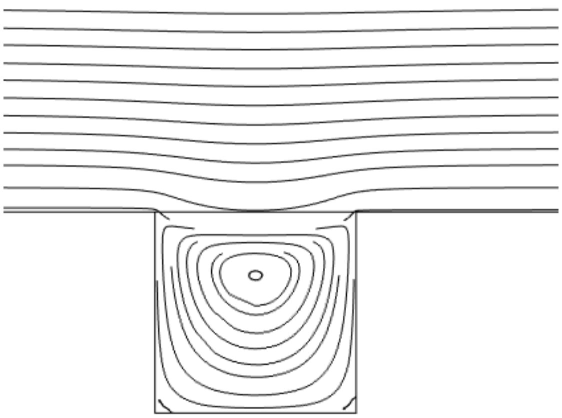
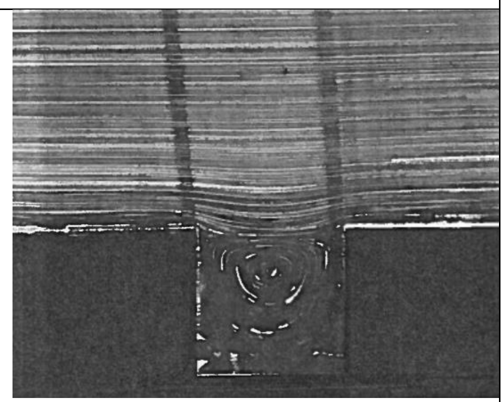
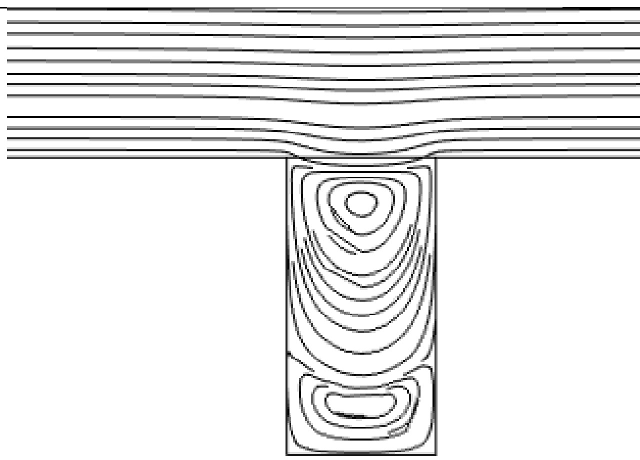
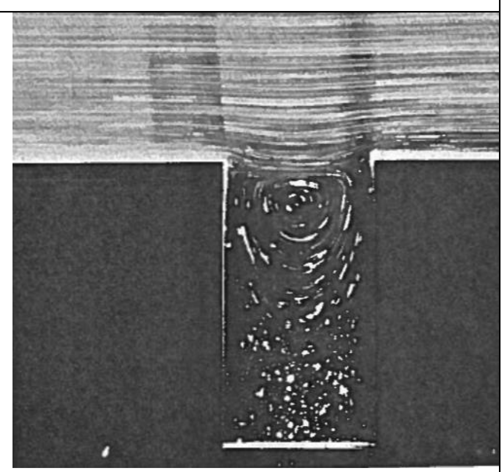
Problem #1

b/h= 3		

b/h= 2		
b/h= 1		
b/h= 0.5		

Agreement looks pretty solid- at least based on the streamlines.

Problem #2

Here I created a 2D channel and placed a small "particle" in the flow. I changed the vertical location of the particle from the top to lower boundary. I then computed the total vertical force on the particle and obtained the result shown below. For a particle between 0.6 units of height and the wall (at 1 unit) feels a force pushing toward the center. For a particle located between the centerline and 0.5 units, the particle feels a push away from the center. The particle in this flow, would thus migrate to an equilibrium point around 0.55 units of height. The analysis assumes the particle is still and going at the local fluid velocity. In reality the speed of the particle is such that it is "force free" – which we could compute by computing the drag force on the object and then adjusting its speed relative to the flow until the x-component of the force is zero. Also the particle is subject to shear and would thus rotate (quite rapidly actually!). A more complete analysis with rotation shows that the equilibrium positions are not changed much if rotation is taken into account.

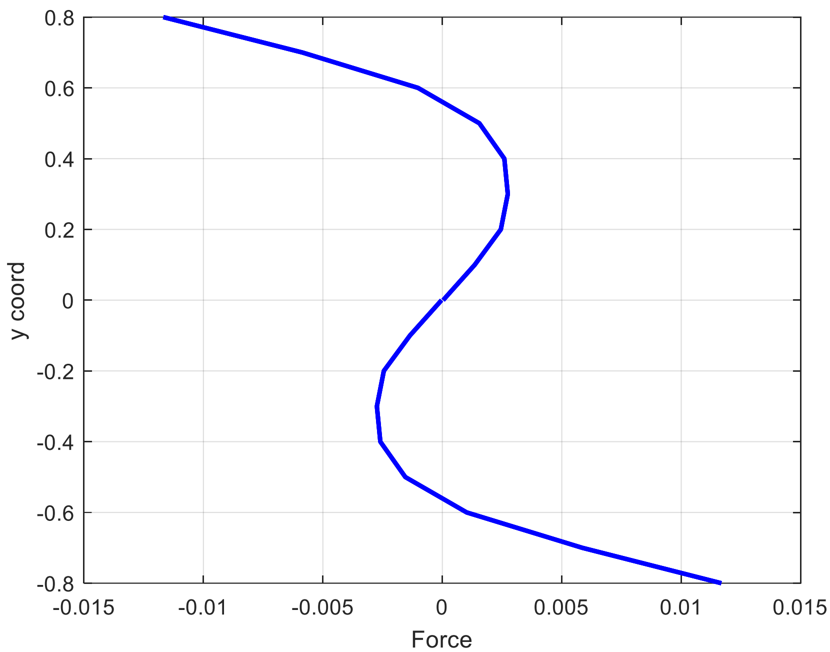


Figure 1: Vertical force on a small cylinder in Poiseuille flow at channel Reynolds number of 100. The "particle" has a vertical equilibrium position a little above channel locations of 0.6. Note that the center location has zero net force, but the particle is unstable at this location.

#1

$$T = \begin{bmatrix} -P & 0 \\ 0 & -P \end{bmatrix} + \mu \begin{bmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & 2\frac{\partial v}{\partial y} \end{bmatrix}$$

$u(y) = 1 - y^2$

$\mu = 1$ since dimensionless

$$T = \begin{bmatrix} -P & 0 \\ 0 & -P \end{bmatrix} + \begin{bmatrix} 0 & \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -P & 0 \\ 0 & -P \end{bmatrix} + \begin{bmatrix} 0 & -2y \\ -2y & 0 \end{bmatrix} = \begin{bmatrix} -P & -2y \\ -2y & -P \end{bmatrix}$$

at $y = -1$ $\quad T = \begin{bmatrix} -P & 2 \\ 2 & -P \end{bmatrix}$

$n = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \vec{n} \cdot T = Tn = \begin{bmatrix} -P & 2 \\ 2 & -P \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ -P \end{bmatrix}}$

since symmetric

X force is viscous shear

Y force is pressure.

at $y=0$ $\quad T = \begin{bmatrix} -P & 0 \\ 0 & -P \end{bmatrix}$

$$n = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \vec{S} = \begin{bmatrix} -P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -P \end{bmatrix}$$

No Shear along Centerline

#2

$u(x,y) = \cos x \sin y$

$v(x,y) = -\sin x \cos y$

$$T = \begin{bmatrix} -P & 0 \\ 0 & -P \end{bmatrix} + \mu \begin{bmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} & 2\frac{\partial v}{\partial y} \end{bmatrix}$$

$$\frac{\partial u}{\partial x} = -\sin x \sin y$$

$$\frac{\partial u}{\partial y} = \cos x \cos y$$

$$\frac{\partial v}{\partial x} = -\cos x \cos y$$

$$\Rightarrow \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial y} = \sin x \sin y$$

Viscous Part of $T = \begin{bmatrix} -2\sin x \sin y & 0 \\ 0 & 2\sin x \sin y \end{bmatrix}$

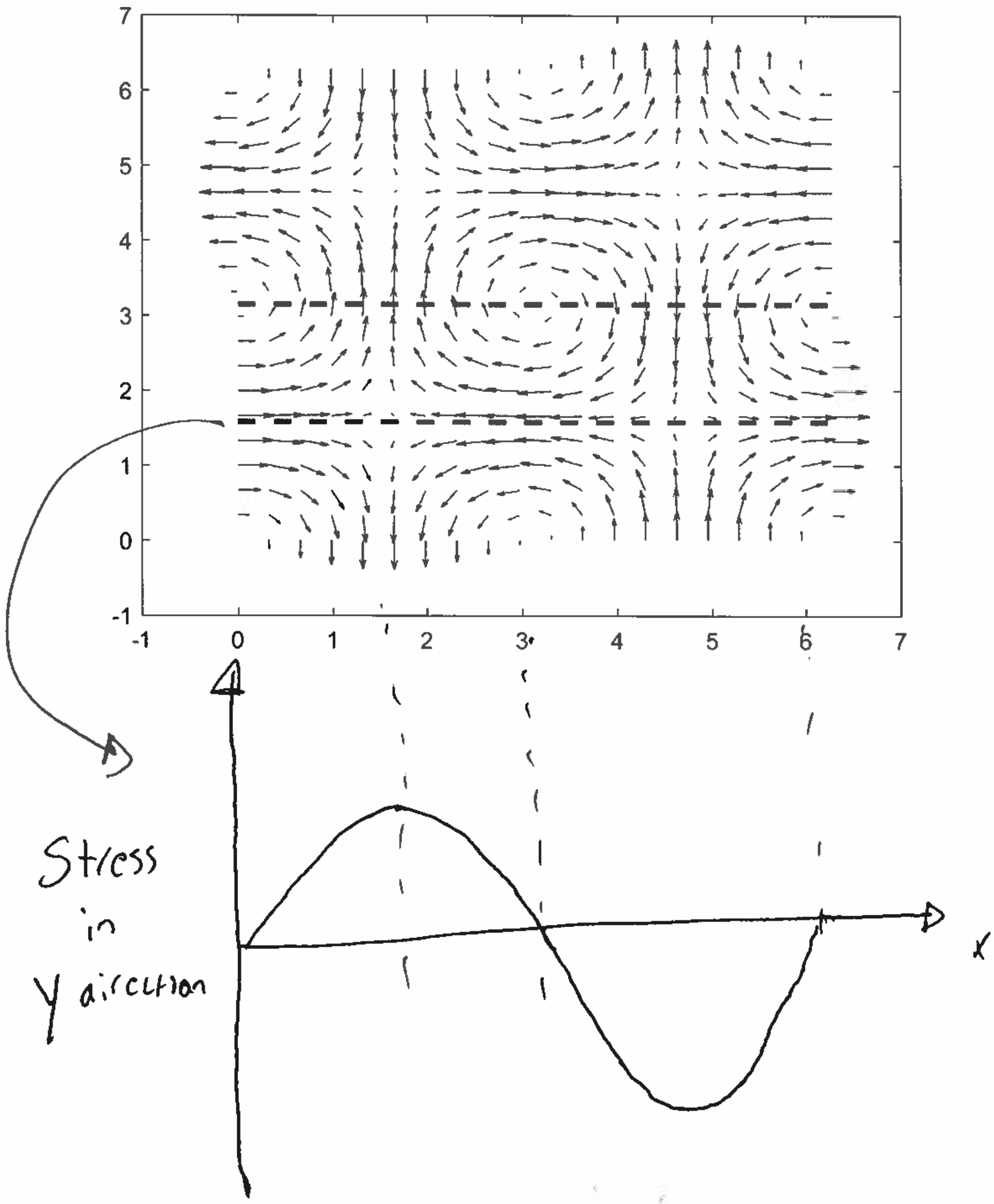
at $y = \frac{\pi}{2}$

$$T = \begin{bmatrix} -2\sin x & 0 \\ 0 & 2\sin x \end{bmatrix}$$

$$n = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \vec{S} = \begin{bmatrix} -2\sin x & 0 \\ 0 & 2\sin x \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2\sin x \end{bmatrix}$$

at $y = \pi$

$$T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$



Stress
in
Y direction
x