## Units

$\nabla \rightarrow \frac{1}{L}$

$\nabla\cdot \rightarrow 1/L$

$dV \rightarrow L^3$

$dS \rightarrow L^2$

$\vec{n} \rightarrow$ No units.

## Integrals

$\int \rho \, d\bar{V}$ = mass of the region of integration [M]

$\int \rho \vec{v}\cdot\vec{n} \, dS$ = total rate that mass is convected across the entire surface $\left[\frac{M}{T}\right]$

$\int T\cdot n \, dS$ = Nonsense

$\frac{1}{V}\int T \, d\bar{V}$ = Average temperature of region of integration

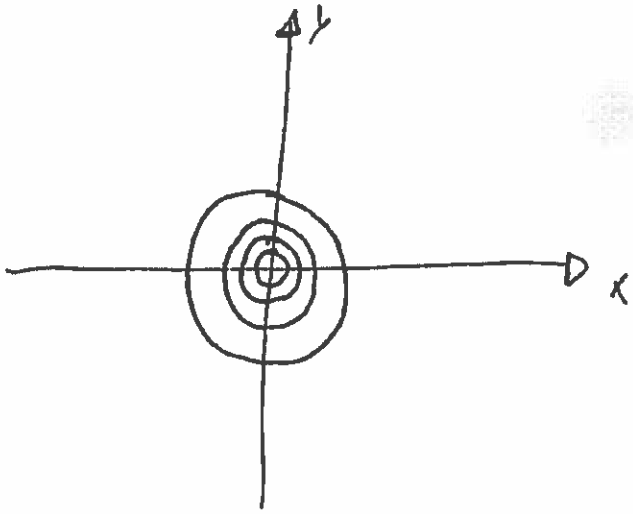
$\int \nabla\cdot(\rho\vec{v}) \, d\bar{V} = \int \rho\vec{v}\cdot\vec{n} \, dS$ = Same as #2 above.

$\int p\vec{n} \, dS$ = total pressure force exerted on surface [Force]

$\int \nabla\cdot\rho \, d\bar{V}$ = Nonsense.

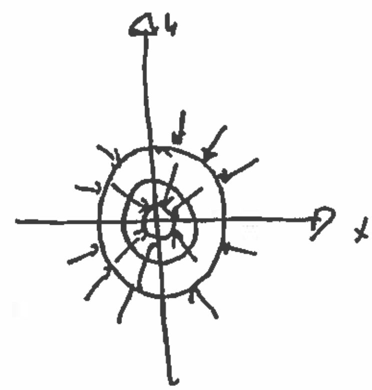
①

$$f = e^{-10(x^2+y^2)}$$




$$\nabla f = -20xf\,\hat{i} - 20yf\,\hat{j}$$

$$= \underbrace{-20f}_{\substack{\Uparrow \\ \text{Mag.} \\ \text{decays} \\ \text{rapidly from origin}}} \underbrace{(x\hat{i} + y\hat{j})}_{\substack{\text{points} \\ \text{radially}}}$$

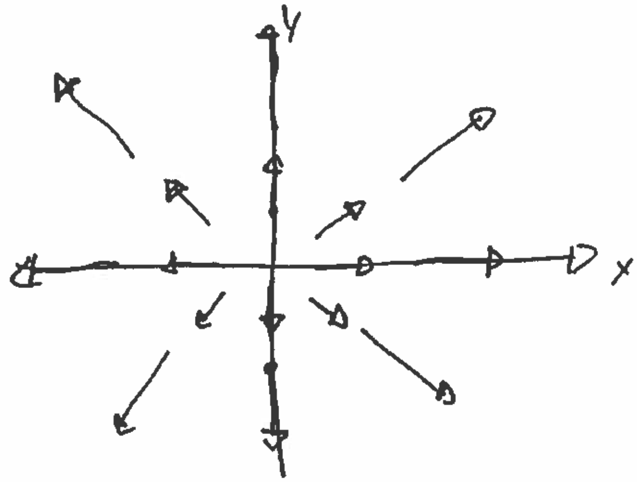



$$\nabla^2 f = (400x^2 - 20)f + (400y^2 - 20f)$$

$$= (400(x^2+y^2) - 40)f$$

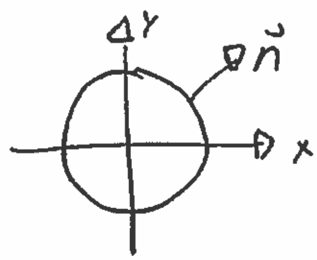
(2)

$f = x\hat{i} + y\hat{j}$




$$\nabla \cdot \vec{f} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 1 + 1 = 2$$

$\vec{f} \cdot \vec{n}$ along circle of radius R

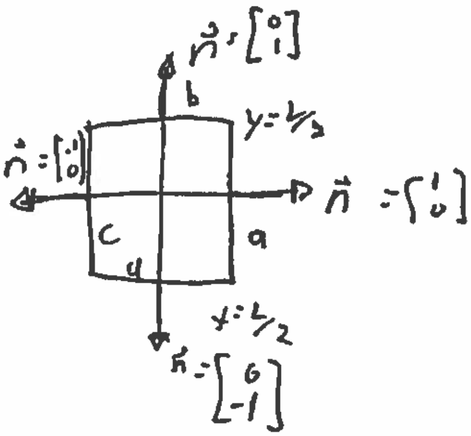



$$\vec{n} = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2+y^2}}$$

$$\vec{f} \cdot \vec{n} = \frac{x^2 + y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} = \underline{\underline{R}}$$

$$\int \vec{f}\cdot\vec{n}\, dS = \int_0^{2\pi} \underbrace{R}_{\vec{f}\cdot\vec{n}}\ \underbrace{R\, d\theta}_{dS} = R^2\, 2\pi$$

$$\int \nabla\cdot f\, dV = \int_0^R\int_0^{2\pi} 2\, r\, dr\, d\theta = 2\cdot \pi R^2$$




$$\underbrace{\int_{-L/2}^{L/2} \frac{L}{2}\, dy}_{\text{side "a"}} + \underbrace{\int_{-L/2}^{L/2} \frac{L}{2}\, dx}_{b} + \int_{-L/2}^{L/2} \frac{L}{2}\, dy + \int_{L/2}^{-L/2} -\frac{L}{2}\, dx = 2L^2$$

Note, twice the area, Same as the Circle!

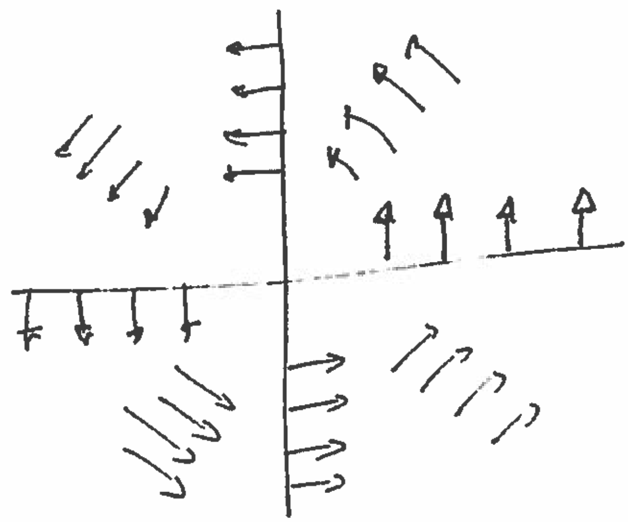
In cylindrical coordinates

$$\vec{F} = r\hat{e}_r$$

$$\nabla \cdot f = \frac{1}{r}\frac{\partial}{\partial r}(r f_r) = \frac{1}{r}\frac{\partial}{\partial r} r^2 = \underline{\underline{2}}$$

③

$$\vec{f} = \frac{-y\,\hat{i} + x\,\hat{j}}{\sqrt{x^2+y^2}}$$



$$\nabla \cdot \vec{f} = \frac{xy}{(x^2+y^2)^{3/2}} - \frac{xy}{(x^2+y^2)^{3/2}} = \underline{\underline{0}}$$

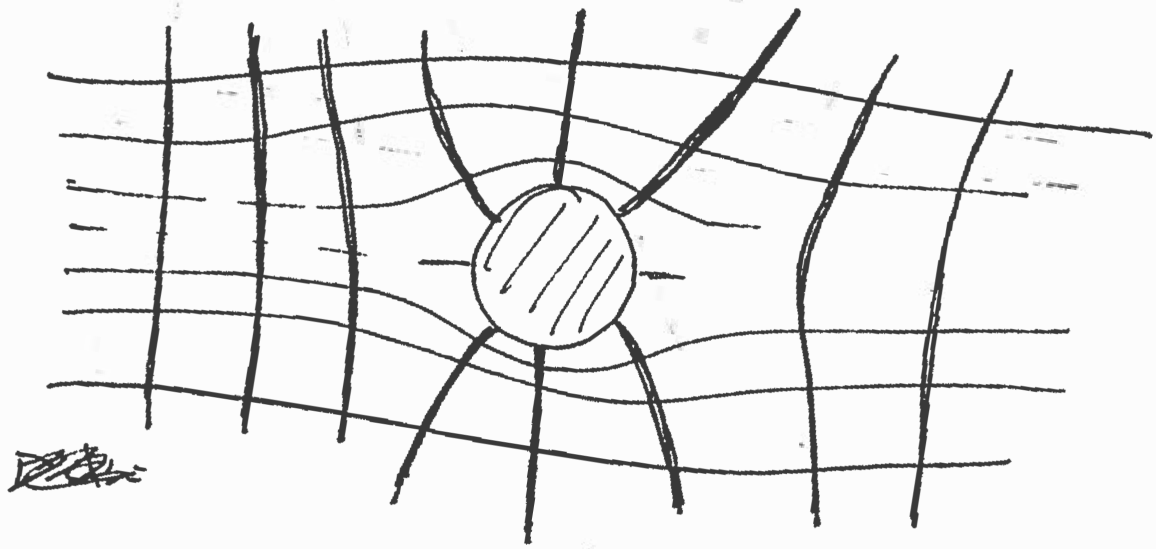
$$\vec{f} \cdot \hat{n} = \frac{-y}{\sqrt{x^2+y^2}} \cdot \frac{x}{\sqrt{x^2+y^2}} + \frac{x}{\sqrt{x^2+y^2}} \cdot \frac{y}{\sqrt{x^2+y^2}} = \underline{\underline{0}}$$

$$\vec{f} = \underline{\underline{\hat{e}_\theta}}$$

$$\nabla \cdot f = \frac{1}{r}\frac{\partial f_\theta}{\partial \theta} = \underline{\underline{0}}$$

⑤

$$\Phi = x + \frac{x}{x^2+y^2}$$



$$\nabla\Phi = \left(1 + \frac{1}{x^2+y^2} - \frac{2x^2}{x^2+y^2}\right)\hat{i} +$$

$$\left(-\frac{2xy}{x^2+y^2}\right)\hat{j}$$

$$\nabla^2\Phi = 0$$

6

$$\nabla = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \qquad \vec{V} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\vec{V} \cdot \nabla = \underbrace{u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}}_{\text{an operator!}}$$

(7)

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix}$$

$\nabla\phi$

Cross product

$$\left(\frac{\partial^2\phi}{\partial y\,\partial z} - \frac{\partial^2\phi}{\partial z\,\partial y}\right)\hat{i} - \left(\frac{\partial^2\phi}{\partial x\,\partial z} - \frac{\partial^2\phi}{\partial z\,\partial x}\right)\hat{j} + \left(\frac{\partial^2\phi}{\partial x\,\partial y} - \frac{\partial^2\phi}{\partial y\,\partial x}\right)\hat{k}$$

$$= 0$$

$$\vec{V} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\nabla \times \vec{v} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{bmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} - \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\hat{j} +$$

$$\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

$$\nabla \cdot (\nabla \times \vec{v}) = \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial z} - \frac{\partial^2 w}{\partial y \partial x} + \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 v}{\partial z \partial x} - \frac{\partial^2 u}{\partial z \partial y}$$

Check $\Rightarrow$ All terms Cancel

$$\underline{\underline{\nabla\left(\tfrac{1}{2}\vec{V}^2\right)}} = \frac{\partial}{\partial x}\left(\tfrac{1}{2}u^2 + \tfrac{1}{2}v^2\right)\hat{i} + \frac{\partial}{\partial y}\left(\tfrac{1}{2}u^2 + \tfrac{1}{2}v^2\right)\hat{j}$$

$$= \left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)\hat{i} + \left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right)\hat{j}$$

$$\underline{\underline{(\vec{V}\cdot\nabla)}}\vec{V} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)u\hat{i} + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)v\hat{j}$$

$$= \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)\hat{i} + \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)\hat{j}$$

$$\nabla\times\vec{V} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

$$\underline{\underline{\vec{V}\times(\nabla\times\vec{V})}} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ u & v & w \\ 0 & 0 & \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{bmatrix} =$$

$$\left(v\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial y}\right)\hat{i} - \left(u\frac{\partial v}{\partial x} - u\frac{\partial u}{\partial y}\right)\hat{j}$$

So

$$(\vec{V}\cdot\nabla)\vec{V} + \vec{V}\times(\nabla\times\vec{V}) = \left(u\frac{\partial u}{\partial x} + \cancel{v\frac{\partial u}{\partial y}} + v\frac{\partial v}{\partial x} - \cancel{v\frac{\partial u}{\partial y}}\right)\hat{i}$$

$$+ \left(\cancel{u\frac{\partial v}{\partial x}} + v\frac{\partial v}{\partial y} - \cancel{u\frac{\partial v}{\partial x}} + u\frac{\partial u}{\partial y}\right)\hat{j}$$

$$= \left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)\hat{i} + \left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right)\hat{j}$$

$$= \left(\frac{\partial}{\partial x}\left(\tfrac{1}{2}u^2 + \tfrac{1}{2}v^2\right)\right)\hat{i} + \left(\frac{\partial}{\partial y}\left(\tfrac{1}{2}u^2 + \tfrac{1}{2}v^2\right)\right)\hat{j}$$

$$= \nabla\left(\tfrac{1}{2}\vec{V}^2\right)$$