Vector calculus

I. READING AND LECTURES

To support this homework

- Read chapter 4 from the book.
- Watch the two YouTube video lectures on vector calculus.

In the following dS is the differential element of surface, dV is the differential element of volume, and $\mathbf{n}$ is the normal vector.

II. UNITS

What are the units of

1. ∇
2. $\nabla\cdot$
3. dV
4. dS
5. $\mathbf{n}$

III. INTEGRALS

In the following, ρ, is the mass density, $\mathbf{v}$ is the velocity vector, P is pressure, and T is temperature. Answer the following questions conceptually, in words. What does the integral mean? Provide units for your answer. Also, a few of the integrals below make no physical or mathematical sense. Note which ones are nonsense by cross them out.

6. $\int \rho dV =$
7. $\int \rho \mathbf{v} \cdot \mathbf{n} dS =$
8. $\int T \cdot \mathbf{n} dS =$
9. $\frac{1}{V} \int T dV =$
10. $\int P \mathbf{n} dS =$
11. $\int \nabla \cdot (\rho \mathbf{v})\, dV =$
12. $\int \nabla \cdot \rho dV =$

IV. COMPUTE

1. Consider the scalar field,

$$f = \exp(-10(x^2 + y^2))$$

   - Sketch by hand the contours of the scalar field on the domain $-1 < x < 1$, $-1 < y < 1$.
   - Compute the gradient and sketch the vector field ∇f. You can use wolfram alpha.
   - Sketch by hand quivers of the vector field, ∇f.

2. Consider the vector function given by,

$$\mathbf{f} = x\mathbf{i} + y\mathbf{j}$$

   - Sketch by hand the vector field.
   - Compute the divergence.
   - Consider a circle of radius R centered at the origin; compute $\mathbf{f} \cdot \mathbf{n}$.
   - Evaluate $\int \mathbf{f} \cdot \mathbf{n} dS$ along the same circle.
   - Evaluate $\int \nabla \cdot \mathbf{f} dV$ for the same circle.
   - Consider a square of side length L centered at the origin, $\int \mathbf{f} \cdot \mathbf{n} dS$ along the boundary of the square.
   - Convert the vector field to one described in cylindrical coordinates.
   - Look up the divergence operator in cylindrical coordinates and recompute the divergence of the field.

3. Consider the vector function given by,

$$\mathbf{f} = \frac{-y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2 + y^2}}$$

   - Plot the vector field.
   - Compute the divergence.
   - Consider a circle of radius R centered at the origin; compute $\mathbf{f} \cdot \mathbf{n}$.
   - Convert the vector field to one described in polar coordinates.
   - Look up the divergence operator in polar coordinates and recompute the divergence of the field.

4. Consider the scalar field,

$$\phi = x + \frac{x}{x^2 + y^2}$$

   This field represents the ideal velocity potential for fluid flow around a cylinder in 2D. The ideal velocity potential assumes that the fluid viscosity is 0 - which can only be realized in the superfluid state with helium near zero Kelvin. While the ideal field cannot be realized in practice, this velocity field does play an important role in airfoil theory (this will come back later in the course). The velocity vectors are defined as $\nabla \phi = \mathbf{v}$. Investigate the field on the domain $-5 < x < 5$, $-5 < y < 5$ and where $x^2 + y^2 > 1$

   - Plot the contours of the scalar field.
   - Compute the gradient and plot the vector field $\nabla \phi$.
   - Show that the potential satisfies the equation, $\nabla^2 \phi = 0$.

5. The **operator** $(\mathbf{v} \cdot \nabla)$ where $\mathbf{v}$ is the vector field of the fluid velocity will show up a lot in our equations in this course. Expand the operator out in component form where the components of velocity are $\mathbf{v} = [u, v, w]$ for the x, y, and z directions. Note that order matters and $(\mathbf{v} \cdot \nabla)$ does not equal $\nabla \cdot \mathbf{v} \cdot$.

6. Demonstrate that

   - $\nabla \times (\nabla \phi) = 0$ where ϕ is a any scalar field. Do this in 3D. You can just carry the operation in component form and show that the result is always zero for any ϕ.
   - $\nabla \cdot (\nabla \times \mathbf{v}) = 0$ where $\mathbf{v}$ is any velocity vector field. Do this in 3D. You can just carry the operation in component form and show that the result is always zero.