HW 4: heat conduction

I. ESTIMATION AND UNITS

1. Estimate the thermal energy contained in a fresh cup of coffee. Convert this equivalent amount of energy into a simple example that you can remember related to kinetic or potential energy. This estimate is just for you to have a useful thinking tool to remember whether an energy change is a "little" or a "lot". This estimate should highlight the difference between mechanical and thermal energy.

2. An aluminum object approximately 5 cm in diameter is heated to 300 C. The object is plunged in a well-stirred water bath at 25 C to quench it. Estimate how long it would take the object to cool - order of magnitude (0.1 sec, 1 sec, 10 sec, 100 sec, etc).

3. If you go deep enough in the earth, the soil temperature is equal to the yearly average temperature of that region. Explain how you can estimate the depth you need to go where the temperature doesn't change throughout the year - assuming you know the soil properties. Look up typical soil properties and compute the depth. Again, this is an estimate for order of magnitude only.

II. DERIVATION WITH HEAT GENERATION

1. The general local version of the heat equation for constant thermal conductivity is

$$\rho C \frac{\partial T}{\partial t} = k \nabla^2 T$$

   Consider a case, such as in an electrical resistor, where in addition to heat conduction there is thermal energy generated *inside* the material volumetrically. Re-derive the heat equation for the case, where heat is generated inside the material uniformly at a volumetric rate of Q (W/m^3).

2. For the above problem, consider the 1D case where the left edge at $x = 0$ is held fixed at T_{cold} and the right edge at $x = L$ is held fixed at also at T_{cold}. Consider the system reaches steady state such that $\partial T/\partial t = 0$. What is steady state $T(x)$?

3. Based on dimensional analysis, what is the important dimensionless number that could tell us something about what the final center temperature might be - without solving the problem?

III. FORMULATION

1. A 1D wall of thickness L, conductivity k, density ρ, and specific heat C, separates a hot and cold region. Imagine a case where initially the wall is cold and at uniform temperature. One side of the wall is then brought instantly to T_{hot} and the other is held at T_{cold} (the initial temperature). Write the equation, boundary conditions, and initial condition for the problem for computing the wall temperature distribution as a function of time. Don't solve, just state the problem mathematically. Sketch the solution, qualitatively.

2. A wall of thickness L, conductivity k, density ρ, and specific heat C, separates the indoors from the outdoors. Imagine a case where initially the indoor air, the wall, and the outdoor air are all at the same temperature, T_{out}. The heat to the building is turned on such that the air inside increases suddenly to T_{in}. Heat is transferred at the wall on the inside and outside by convection, with coefficient h_{in} and h_{out}. Write the equation, boundary conditions, and initial condition for computing the wall temperature distribution as a function of time. Don't solve, just state the problem mathematically. Sketch the solution, qualitatively.

IV. SOLUTIONS AT STEADY STATE

1. Consider the previous problem of the 1D wall with inside and outside convection. At **steady state** when $\partial T/\partial t = 0$, solve for the temperature field inside the wall. The result should be in terms of known quantities, such as the inside and outside air temperatures, the inside and outside convection coefficients, and the length

of the wall. Sketch the temperature field for a few cases. Consider the cases where the inside and outside convection coefficients are equal and large, as well as when they are equal and small. Try to find an important dimensionless parameter embedded in your solution.

2. For the previous problem at steady state, solve for the overall thermal resistance which accounts for indoor convection, conduction through the wall, and outdoor convection. The overall thermal resistance is that which follows, $T_{in} - T_{out} = qR$, where q is the heat flow (W/m^2).

3. Consider a wall of thickness L. From $0 < x < L/2$ the conductivity of the material is k and from $L/2 < x < L$ the conductivity is $10k$. Consider that initially the temperature is T_c. At $t > 0$, $T(x = 0, t) = T_h$ and $T(x = L, t) = T_c$. Sketch the steady state temperature profile. Assuming the two regions have the same density and specific heat, ρ and C, estimate how long it will take for the temperature field to come to steady state? Sketch the temperature profile as a function of time.

V. PLOT SOME SOLUTIONS

1. Consider heat flow in a semi-infinite domain. The domain spans $0 < x < \infty$. Initially the temperature everywhere is T_0. At $t = 0$ the temperature at the boundary is suddenly increased to $T(x = 0, t) = T_h$. The solution to the dimensional problem is

$$\frac{T(x,t) - T_0}{T_h - T_0} = 1 - \mathrm{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right).$$

The function erf is known as the error function. You can look up the definition on wikipedia. First, use Wolfram alpha to confirm that the solution above satisfies the heat equation. Plot the solution in dimensionless (just the equation above) and dimensional terms for the problem of soil in the earth responding to a sudden change in temperature at the surface. Look up reasonable properties for the thermal diffusivity of soil (from the first section) and show the evolution for 6 months. While the more realistic problem for the soil would have periodic forcing, the step response is just easier for us to write down the solution. Compare the results of this solution to your estimation from the first section.

2. Consider the following dimensionless heat conduction problem,

$$\frac{\partial \Theta}{\partial t} = \frac{\partial^2 \Theta}{\partial x^2}$$

with initial condition $\Theta(x, t = 0) = 1$ and boundary conditions $\Theta(x = 0, t) = 0$ and $\Theta(x = 1, t) = 0$. The analytical solution to this problem is

$$\Theta(t, x) = \sum_{n=1,3,5,..}^{\infty} \frac{4}{n\pi} \mathrm{e}^{-n^2\pi^2 t} \sin(n\pi x)$$

- Show that this function satisfies the equation.
- Show that this function satisfies the boundary conditions.
- Show that this function satisfied the initial condition (which is easiest to demonstrate numerically).
- Calculate the heat flux at $x = 0$ as a function of time and write the result as a sum.