Vector calculus

1. Consider the scalar field,

$$f = \exp(-10(x^2 + y^2))$$

   - Sketch by hand the contours of the scalar field on the domain $-1 < x < 1$, $-1 < y < 1$.
   - Compute the gradient and sketch the vector field ∇f. You can use wolfram alpha.
   - Compute the Laplacian ∇^2. You can use wolfram alpha.

2. Consider the vector function given by,

$$\mathbf{f} = x\mathbf{i} + y\mathbf{j}$$

   - Sketch by hand the vector field.
   - Compute the divergence.
   - Consider a circle of radius R centered at the origin; compute $\mathbf{f} \cdot \mathbf{n}$.
   - Evaluate $\int \mathbf{f} \cdot \mathbf{n} dS$ along the same circle.
   - Evaluate $\int \nabla \cdot \mathbf{f} dV$ for the same circle.
   - Consider a square of side length L centered at the origin, $\int \mathbf{f} \cdot \mathbf{n} dS$ along the boundary of the square.
   - Convert the vector field to one described in cylindrical coordinates.
   - Look up the divergence operator in cylindrical coordinates and recompute the divergence of the field.

3. Consider the vector function given by,

$$\mathbf{f} = \frac{-y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2 + y^2}}$$

   - Plot the vector field.
   - Compute the divergence.
   - Consider a circle of radius R centered at the origin; compute $\mathbf{f} \cdot \mathbf{n}$.
   - Convert the vector field to one described in polar coordinates.
   - Look up the divergence operator in polar coordinates and recompute the divergence of the field.

4. Consider the vector function given by,

$$\mathbf{f} = -x\mathbf{i} + y\mathbf{j}$$

   - Sketch the vector field.
   - Compute the divergence.
   - Consider a square of of side L centered at the origin; compute $\mathbf{f} \cdot \mathbf{n}$.
   - Evaluate $\int \mathbf{f} \cdot \mathbf{n} dS$ along the same square.

5. Consider the scalar field,

$$\phi = x + \frac{x}{x^2 + y^2}$$

   This field represents the ideal velocity potential for flow around a cylinder in 2D. The ideal velocity potential assumes that the fluid viscosity is 0 - which can only be realized in the superfluid state near zero Kelvin. While the ideal field cannot be realized in practice, this velocity field does play an important role in airfoil theory (I can explain in class if you are interested). The velocity vectors are defined as $\nabla\phi = \mathbf{v}$. Investigate the field on the domain $-5 < x < 5$, $-5 < y < 5$ and where $x^2 + y^2 > 1$

   - Plot the contours of the scalar field.

- Compute the gradient and plot the vector field $\nabla\phi$.
- Show that the potential satisfies the equation, $\nabla^2\phi = 0$.

6. The operator $(\mathbf{v} \cdot \nabla)$ where $\mathbf{v}$ is the vector field of the fluid velocity will show up a lot in our equations in this course. Expand the operator out in component form where the components of velocity are $\mathbf{v} = [u, v, w]$ for the x, y, and z directions.

7. Demonstrate that

- $\nabla \times (\nabla\phi) = 0$ where ϕ is a scalar field. Do this in 3D.
- $\nabla \cdot (\nabla \times \mathbf{v}) = 0$ where $\mathbf{v}$ is a velocity vector field. Do this in 3D.
- $\nabla\left(\frac{1}{2}\mathbf{v}^2\right) = (\mathbf{v} \cdot \nabla)\mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{v})$. Do this one in 2D for simplicity.