Solutions to Navier-Stokes

In the following problems we will make use of the simple solutions to the Navier Stokes where we assume long channels where the flow doesn't change along the length of the channel. In this case, the flow velocity only changes across the channel. In all the problems here we construct the solution as the sum of Couette and Poiseuille flow.

1. We will demonstrate a simple experiment in class of a solid cylinder dropping through a liquid filled tube. Outside of class, we recorded several data points which are reported in Table 1. In the experiment we can vary different parameters such as the fluid and the geometry.

D_{cyl} (mm)	L_{cyl} (mm)	M_{cyl} (g)	D_{tube} (mm)	ρ (kg/m^3)	μ (Pa-s)	V (cm/s)
24.43	100	66.4	25.8	0.0009	1000	2.05
23.34	100	61.2	25.8	0.0009	1000	7.4
22.43	100	56.1	25.8	0.0009	1000	14.2
24.38	200	132.5	25.8	0.0009	1000	2.05
24.43	100	66.4	25.8	0.0495	1210	0.042
23.34	100	61.2	25.8	0.0495	1210	0.177
24.38	100	56.1	25.8	0.0495	1210	0.412

TABLE I Experimental data for a cylinder dropping in a tube.

   Using the Navier-Stokes equations, develop an analytical solution for the problem and compare this solution to the experimental data. Yes, I know that this problem description is vague. I would like you to observe the experiment, and try to explain the behavior you see. This exercise is closer to how a "real" problem is stated rather than the usual book problems.

2. Normally in fluid dynamics we assume the no-slip condition at a solid wall. This condition is empirically true for most cases, though there are exceptions where slip has been observed. An interesting case where there is "apparent" slip is when you have charged walls, ions in the fluid (like a dissolved salt solution), and applied electric fields.

   A material like glass will typically become negatively charged when the solid comes in contact with water. This negatively charged wall will attract positive ions from the aqueous salt solution to screen the surface charge. However, molecular thermal motion will keep the ions fluctuating such that a diffuse layer of positive charge will form in the liquid. This layer will have a thickness on the order of 10 nm (that's nanometers) for a 1 mM (milli-molar) salt solution, the length scale is called the Debye length. More than 10 nm away from the wall, the solution will be neutral with the number of negative ions balanced by the number of positive.

   In this case, if you apply an axial electric field down the channel, the electric field will exert a force on the positively charged ions near the wall and the resulting force will drag the fluid along. This electrical force will only be exerted in this charged layer which is 10 nm thick. Thus, even though the fluid at the wall will obey the no slip condition, the fluid 10nm away will be in motion. The net result after is that the flow in a channel which is much larger than 10 nm will have an apparent slip velocity at the wall. This slip velocity is proportional to the electric field. The formula for the slip is $U_{slip} = -\epsilon_0 \epsilon_w \zeta E/\mu$, where ϵ_0 is the vacuum permittivity, ϵ_w is the relative permittivity for water (approximately 80), E is the applied electric field, μ is the dynamic viscosity, and ζ is the so-called zeta-potential, the voltage difference between the wall and the solution far away. In typical glass/water systems $\zeta \approx -100$ mV. This theoretical result is over 100 years old, it is called the Helmholtz-Smoluchowski equation. The effect, called electroosmosis has been known experimentally for over 200 years and was first observed in porous clays. Sometimes it is more convenient to measure the combination of parameters and call it the electroosmotic mobility, $b = -\epsilon_0 \epsilon_w \zeta/\mu$, and simply express the slip velocity as $U_{slip} = bE$.

   If we have a long, thin tube with no applied pressure difference and the electroosmotic (EO) slip velocity is a constant, you would find trivially, that the solution to the Navier-Stokes equation is a constant fluid velocity. All the fluid moves down the tube at the slip velocity. In recent years, these electroosmotic flows have been exploited in a number of modern micro and nanoscale systems. Given the results you have already found for pressure driven flow, you saw that the hydraulic resistance scales poorly as the channel or pipe becomes very small. Thus in small systems, electroosmosis is a practical and efficient pump. It is also a simple pump because it has no moving parts, just apply a voltage across your channel.

While all this background may sound complicated, the final result is quite simple. From the system parameters we can compute the slip velocity. We then solve the flow problem using the slip velocity as the boundary condition for the Navier Stokes. At a solid surface, the normal component of the velocity is zero while the tangential velocity is given as the slip value.

OK. That was all background. Now the actual problems.

- Consider a small tube of radius r and length L. Assume a pressure ΔP and voltage ΔV difference are applied across the tube. The tube is long such that we can consider the flow invariant in the axial direction. The pressure gradient $dP/dx = -\Delta P/L$ and the electric field $E = -\Delta V/L$ are constants along the length of the channel. We will assume the electroosmotic mobility is a known constant (usually comes from measurement). Thus, the apparent slip velocity at the wall has a known value U_{slip}. Solve the Navier Stokes equations for this case. State the velocity profile as a function of r and how it depends on the system parameters, $r, L, \Delta P, \Delta V, \mu$, and U_{slip}. Compute the average fluid velocity i.e. $U_{ave}\pi r^2 = Q$ where Q is the total volumetric flow rate. Write your expression for the average velocity in the form, $U_{ave} = A\Delta P + B\Delta V$, where A and B are constants that you will figure out. If you are really smart you can use superposition and look at the solution for pressure driven flow in a pipe (section 8.9 of the book).
- Consider a tube of radius 10 microns and length 10 mm, containing an aqueous solution. The measured mobility is $b = 7 \times 10^{-8} \frac{\text{m}^2}{\text{V}\cdot\text{s}}$ and the electric field is in Volts/m. The applied voltage is 5 kV. Compute the mean flow velocity if there is no applied pressure gradient. Compute the applied pressure needed to obtain the same mean flow velocity if there were no applied electric field. Repeat the above calculations if the tube has a radius of 1 micron.
- In some applications, we can intentionally coat or modify the solid surface to change the surface charge and thus change the apparent slip velocity. Consider a case where the channel from $0 < x < L_1$ has slip velocity $U_{s,1}$ and $L_1 < x < L$ has slip velocity $U_{s,2}$. Consider flow in such a system where we have two regions of two different slip velocities, but NO applied pressure. Since the channel is long compared to the other scales, we can assume everything is uniform in the axial direction and simply assume we have two channels which are connected together in series.

  Since the slip velocity is different in the two regions, but the total flow through any cross section of the channel must be the same, something has to give. What happens is that while the net applied pressure is zero, each region will have to acquire an internally generated pressure gradient. This pressure gradient will enhance or retard the electrically driven slip flow such that each region has the same total flow rate. The pressure gradient in each region is a constant and described as $dP/dx|_1 = -\Delta P_1/L_1$, and $dP/dx|_2 = -\Delta P_2/L_2$ where the total applied pressure is zero, $\Delta P_1 + \Delta P_2 = 0$. Due to continuity of electrical current, the electric field is a constant in both regions, $E = E_1 = E_2 = \Delta V/L$. The average flow in both regions must be equal, $U_{ave,1} = U_{ave,2}$. Use all the above relationships as well as your formula from part 1, $U_{ave} = A\Delta P + B\Delta V$, to determine the pressure gradient in each region.
- Assume the following parameters. A tube of radius 10 microns and total length 10 mm. The electroosmotic mobility is measured to be $b = 7 \times 10^{-8} \frac{\text{m}^2}{\text{V}\cdot\text{s}}$ in region 1. In region 2 it is reduced to $b = 0.7 \times 10^{-8} \frac{\text{m}^2}{\text{V}\cdot\text{s}}$. The applied voltage is 5000 Volts. Consider cases for $L_1/L_2 = 1/4$, 1, and 4. Compute the average fluid velocity in both regions for all cases of L_1/L_2. Note that the average velocity in region 1 and region 2 better be equal in each case. Check your result and make sure that the limiting cases work, i.e. L_1 and L_2 going to zero limit to the result you derived in part 1.

3. Read through the slider bearing problem in section 8.6 of the book.

   Now consider that the block has three regions of height and the total length is $3L$. From $0 < x < L$ and $2L < x < 3L$ the height is h. In the middle from $L < x < 2L$ we will consider two cases where the height is either $h/2$ or $2h$. Assume the total pressure difference across the block is zero, as in the problem in section 8.6.

   - Without writing a single equation, sketch the velocity field under the block and pressure as a function of length for this problem. This plot need not be quantitative. Try to reason whether the block experiences a force perpendicular to the wall, and if so is it attractive or repulsive?
   - Compare your educated guesses to a 2D steady state Comsol simulation. Note when doing the Comsol simulation you need to shift the reference frame so you are riding on the block and the wall below you is moving. For boundary conditions you will make the block surface a no-slip wall, the lower flat wall of the channel a moving wall, and the two ends should be outlets with zero pressure (check normal flow, and uncheck suppress back-flow).

- Construct the analytical solution to this problem by using the results derived in section 8.6. You can confirm your results by comparing to the comsol simulation.