Boundary layers, drag, and control volumes

I. DRAG ESTIMATES

The following problems are estimates only.

1. How much power is lost to drag with a typical car going 55 mph? 75 mph? If drag were the only loss, what would the top speed be based on the power output of the engine?

2. A reporter is standing outside during a hurricane. How high does the wind speed need to be before the reporter cannot stand normally (straight up with both feet together) ?

II. BOUNDARY LAYERS

1. A passenger plane is flying at cruise altitude of 30,000 ft. How far from the leading edge of the wing does the boundary layer become turbulent (if at all) ? Considering the flow along the fuselage, what is the boundary layer thickness on the fuselage near the back of the plane? This is an estimate.

2. For air flowing at a meter per second over a flat plate, how far down does the flow transition to turbulence. Make a dimensional plot of boundary layer thickness versus distance to get a feel for the order of magnitude of numbers involved.

III. INTERNAL PIPE FLOW

1. A 1/2 inch pipe has water flowing through it at 10 liters per minute. The pipe is copper which has a typical roughness of 0.002 mm. What is the pressure drop required to drive this flow in a 10 meter length of pipe?

2. If we double the applied pressure in the previous problem, what is the resulting flow rate?

IV. TURBULENT INTERNAL FLOWS

Last week we used Bernoulli's equation to understand the levitation a plate when air is blowing outwards. This week, let's refine out analysis by accounting for turbulent friction.

- **Derive the basic equations**. To make progress, we will simplify the Navier Stokes equations significantly. Start with the Navier-Stokes equations for an incompressible, Newtonian fluid in cylindrical coordinates. Assume flow is only radial, v_r, such that the theta and z components of the velocity are zero. Also assume the problem is symmetric and all theta derivative drop out. The radial momentum equation and conservation of mass with these simplifications are

$$\frac{\partial(rv_r)}{\partial r} = 0,$$

$$\rho v_r \frac{\partial v_r}{\partial r} = -\frac{\partial P}{\partial r} + \mu \frac{\partial^2 v_r}{\partial z^2}.$$

Conservation of mass therefore tells us that

$$v_r = \frac{Q}{2\pi r H},$$

where Q is the total flow rate and H is the gap height. The expression here assumes that v_r is only a function of r or is the average velocity. We can now take this simplified momentum equation and group some terms together to obtain,

$$\frac{\partial(\frac{1}{2}\rho v_r^2)}{\partial r} = -\frac{\partial P}{\partial r} + \mu \frac{\partial^2 v_r}{\partial z^2},$$

or

$$\frac{\partial}{\partial r}\left(\frac{1}{2}\rho v_r^2 + P\right) = \mu\frac{\partial^2 v_r}{\partial z^2}.$$

We can see then that if the viscosity is taken as zero, then we have Bernoulli's equation along a streamline from the center to the outer edge. However, lets leave in viscosity and integrate the equation over the gap height (and assuming there is symmetry between $z = 0$ and $z = H$),

$$\int_0^H \left(\frac{\partial}{\partial r}\left(\frac{1}{2}\rho v_r^2 + P\right)\right) dz = -2\mu \left.\frac{\partial v_r}{\partial z}\right|_{z=0}$$

If we define $\bar{v}_r$ as the appropriately averaged velocity then we have

$$\frac{\partial}{\partial r}\left(\frac{1}{2}\rho \bar{v}_r^2 + P\right) = -\frac{2\mu}{H}\frac{\partial v_r}{\partial z}$$

Note that $\tau = \mu\frac{\partial v_r}{\partial z}$ is the shear stress at the solid surface,

$$\frac{\partial}{\partial r}\left(\frac{1}{2}\rho \bar{v}_r^2 + P\right) = -\frac{2\tau}{H}.$$

If the flow were laminar we could have some hope of solving for the shear stress exactly. However, the flow is certainly turbulent and the shear stress τ is given by the empirical parameter friction factor, f. Empirically, for flow in a pipe we describe the shear stress as $4\tau = \frac{1}{2}\rho v^2 f$, where L is the length and D the diameter. The factor of 4 is just by convention. To simplify things we will assume the friction factor is a constant and will not bother to account for its dependence on Reynolds number.

The resulting equations for conservation of mass and momentum allow us to easily solve for the pressure and radial velocity as a function of radius;

$$\bar{v}_r = \frac{Q}{2\pi r H}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{2}\rho \bar{v}_r^2 + P\right) = -\frac{\rho \bar{v}_r^2 f}{4H}$$

Substituting in the expression for Q we have,

$$\frac{\partial}{\partial r}\left(\frac{1}{2}\rho \bar{v}_r^2 + P\right) = -\frac{\rho Q^2 f}{4H(2\pi H)^2 r^2}$$

- **Solve the equations**. The equations are simple enough that you can integrate them directly and solve for the pressure as a function of radius. The boundary conditions is that the pressure at the outer radius matches that of the atmosphere. You can use Wolfram alpha or some other symbolic manipulator if that helps, but it is easy enough to do by hand.
- **Compare to experiments.** You should now be able to compare what this relatively simple model predicts versus what the experiment says. There are data for two different configurations providing total flow, pressure distribution, and plate height. See last week's homework. You should be able to select a single friction factor that models both data sets reasonably well, though you can use a slightly different factor for each experiment.

V. MOMENTUM AND CONTROL VOLUMES

1. A circular jet of water of diameter D and velocity V strikes a rectangular block as shown. The block has a height H, width W, and a mass M. The jet strikes the block in the center. What is the minimum jet velocity needed to tip the block over.

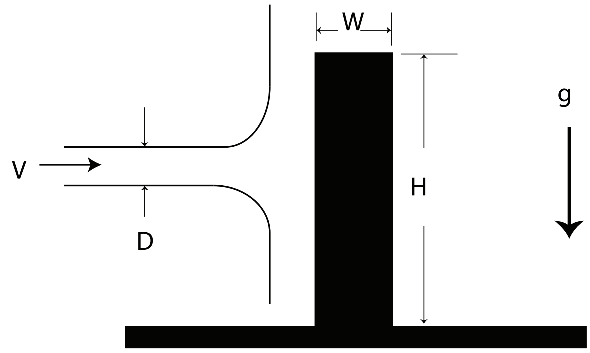

FIG. 1 Schematic for problem 1.

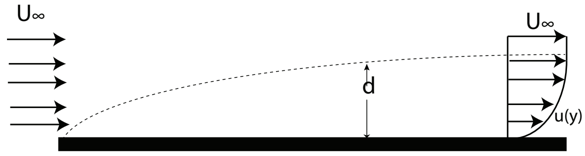

FIG. 2 Schematic for problem 2.

2. Water flows over a flat plate. At the leading edge of the plate there is uniform velocity U_∞. At the trailing edge we measure the velocity boundary later that has developed across the plate, $u(y)$. We measure the velocity as a function of y, the distance from the plate. The boundary layer has the conditions that $u(y = 0) = 0$ and $u(y > d) = U_\infty$. Using an appropriate control volume, measure the drag force on the plate as a function of the free stream velocity, the measured velocity profile, the boundary layer thickness, an any relevant fluid properties. Your answer will be in terms of an integral which you could evaluate numerically if you made the measurement.

3. The hydraulic jump can be observed in the bottom of a sink. Notice that as the water spreads radially, that there is point where the height of the water suddenly increases. This jump can be observed in the spillways of dams and in rivers. The transition from height h_1 upstream can be related to the height h_2 downstream. Assume the upstream velocity, U_1, and the downstream velocity U_2 are constant across the layer depth. Determine the relationship between the upstream and downstream heights. Try to rearrange everything such that $h_2/h_1 = f(Fr)$ where the Froude number is, $Fr = U_1/\sqrt{gh_1}$.

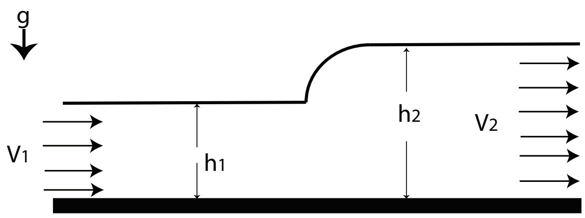

FIG. 3 Schematic for problem 3.