Bernoulli and inviscid flow

I. BERNOULLI

See below for schematics for the following problems.

1. Consider air flow through a venturi meter at a flow rate Q. The cross sectional area of the inlet and contraction are known. The height of water in a manometer is measured. Derive a relationship for the flow rate as a function of the measured height.
2. Air flows through a duct. Determine the velocity of the air as a determined by the measured height in the manometer.
3. From the measurement of the height of water before and after a sluice gate, estimate the total flow rate through the open channel.
4. The inlet to a venturi is at atmospheric pressure. Air flows through the venturi at flow rate Q_a. A small tube is connected to a tank of liquid through a thin pipe. The liquid is drawn into the venturi by the low pressure at the throat. The liquid flow rate, Q_l, can be computed through Poiseueille's law ($\Delta P = Q128\mu L/(\pi D^4)$). Derive a formula for the flow rate of liquid in terms of the flow rate of air, the fluid properties of air and the liquid, the geometry of the venturi, and the length/diameter of the liquid pipe.

II. LEVITATING PLATE

In class I will demonstrate a simple experimental setup. Essentially air blows out of an opening in a flat plate. Another plate is brought close to the opening and this plate is pulled upward and levitates like an upside down air hockey puck, even though the air is blowing outward. The behavior seems counterintuitive, but is easily explained with basic fluid dynamics.

To make progress, we will simplify the Navier Stokes equations significantly. Start with the Navier-Stokes equations for an incompressible, inviscid fluid in cylindrical coordinates. Assume flow is only radial, v_r, such that the theta and z components of the velocity are zero. Also assume the problem is symmetric and all theta derivatives drop out. Note that we will neglect viscosity. The radial momentum equation and conservation of mass with these simplifications are

$$\frac{\partial (r v_r)}{\partial r} = 0,$$

$$\rho v_r \frac{\partial v_r}{\partial r} = -\frac{\partial P}{\partial r}.$$

- Look up the Navier Stokes in cylindrical coordinates and confirm the above equation for our assumptions.
- For the case where a constant volumetric flow rate is forced into the inner hole, solve for the velocity as a function of radial coordinate.
- Use the momentum equation to solve for pressure as a function of radial coordinate.
- Compare your result to the experimental measurements seen in the figure below.
- Your model cannot explain the data where the system is inverted and the plate is on top. Why?

III. VORTEX DYNAMICS

The vorticity, ω, is related to the solid body rotation of infinitesimal fluid particles. The vorticity is given as the curl of the velocity field, $\nabla \times \mathbf{v}$. The 2D ideal vortex has a velocity field in radial coordinates given as

$$v_\theta = \frac{\Gamma}{2\pi r}$$

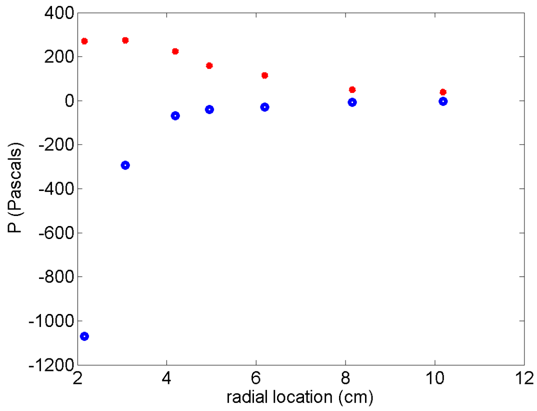

FIG. 1 Pressure data for the levitating plate. The blue data with open circles the device is oriented downward such that the plate is held up with respect to gravity. If the air flow is turned off the plate falls to the ground. For this case the air flow was measured to be 0.0138 m^3/s and the gap height was measured to be 2 mm. The red data with closed dots the device is inverted and the plate is held up with respect to gravity. If the air flow is turned off the plate collapses to the blower. In this case the flow rate was measured to be 0.0023 m^3/s and the gap height was determined to be 0.7 mm.

- Show that the circulation around this vortex is given by Γ.
- Show that, except at the origin of the vortex, the vorticity everywhere else is zero.
- Since the vorticity is zero everywhere except at the origin of the vortex, the velocity field can be shown to follow a potential. When $\nabla \times \mathbf{v} = 0$ we can define the velocity potential, ϕ. The potential is related to the velocity through $\nabla\phi = \mathbf{v}$. Prove that we can define a velocity potential by showing that $\nabla \times \nabla\phi = 0$. You can do this simple vector calculus proof by carrying out the operation component by component.
- Find the velocity potential which gives the ideal vortex velocity field.
- Since the velocity is defined by $\nabla\phi = \mathbf{v}$, show that the equation for fluid motion is $\nabla^2\phi = 0$. What's nice about this result is that the equation is linear, thus when flow has no vorticity we can add up the velocity potentials for different simple flows to get a more complex one. We are allowed to use superposition.

In 2D the vorticity equation is

$$\frac{D\omega}{Dt} = 0$$

The interesting thing about this equation is that it says that vorticity is locked in the fluid and just goes with the flow. Vorticity of individual fluids particles does not change. This means that if we have a single ideal point vortex, it sits there spinning and the vortex cannot move itself. If we have two vortices, since the velocity field away from the origin is irrotational, the velocity field of each vortex adds up everywhere in the fluid. The velocity field of one vortex moves the other at its center and vice-versa. To simulate point vortices, we can write a very simple program using these facts.

We go to each point vortex in the simulation. We compute the velocity of that vortex's center by adding up the velocity field of all other vortices. We can then move the vortex in simulation a small distance using Euler integration, i.e. new position is current position plus velocity times time. We then update all vortex positions and move on. We will review this method in class.

Write a simple program to simulation the motion of vortices. Confirm your simulation with two vortices. The two vortices should move each other in a straight line when given the same strength, but opposite spin directions. The two vortices should orbit each other in a circle if given the same strength and the same direction. If you are feeling good about this, try to extend to four vortices and see if you can simulate the approach of two vortices with a wall as discussed in the book.

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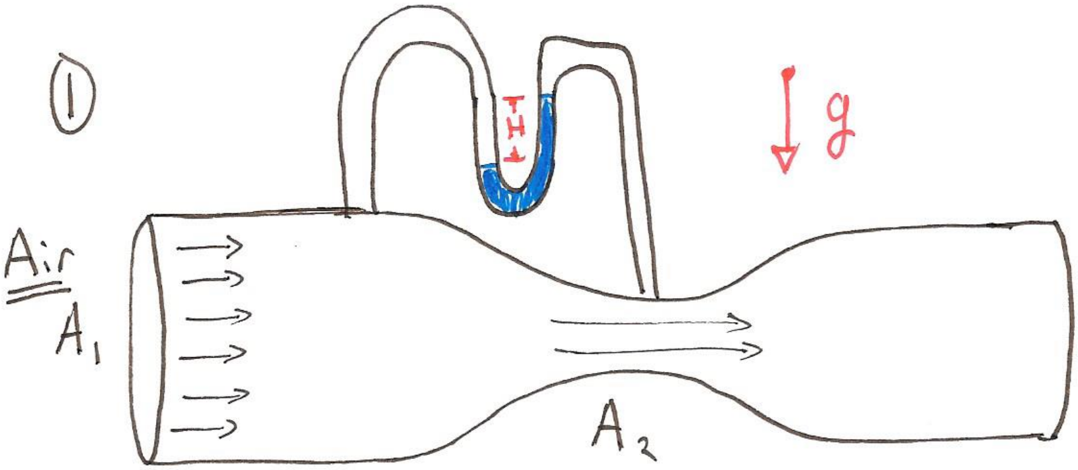
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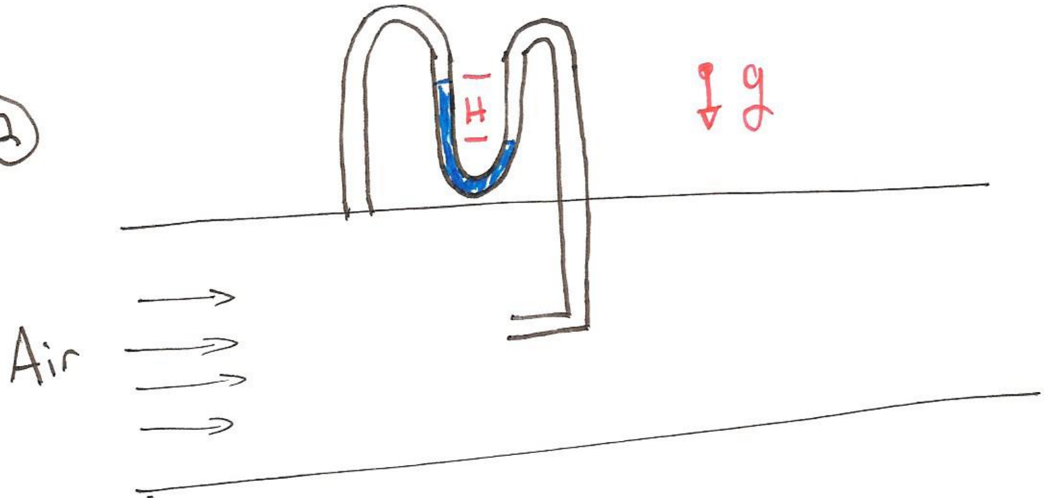
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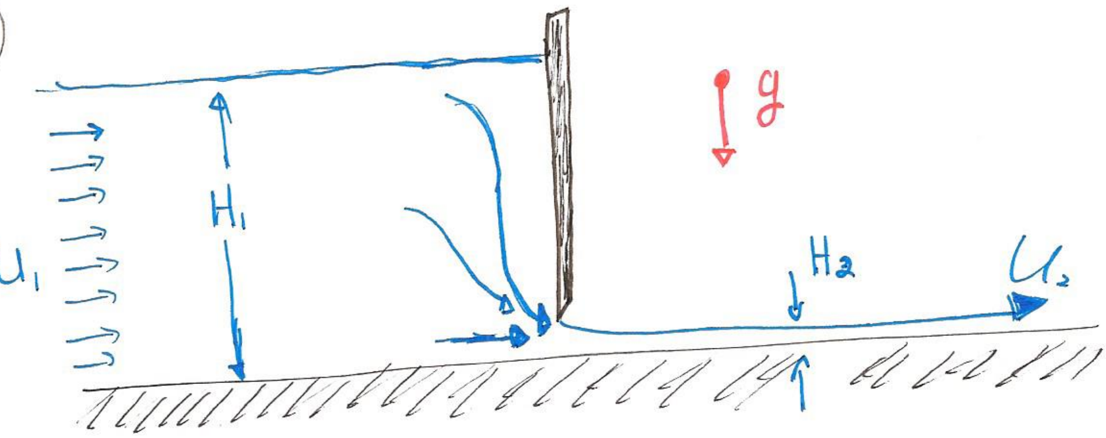
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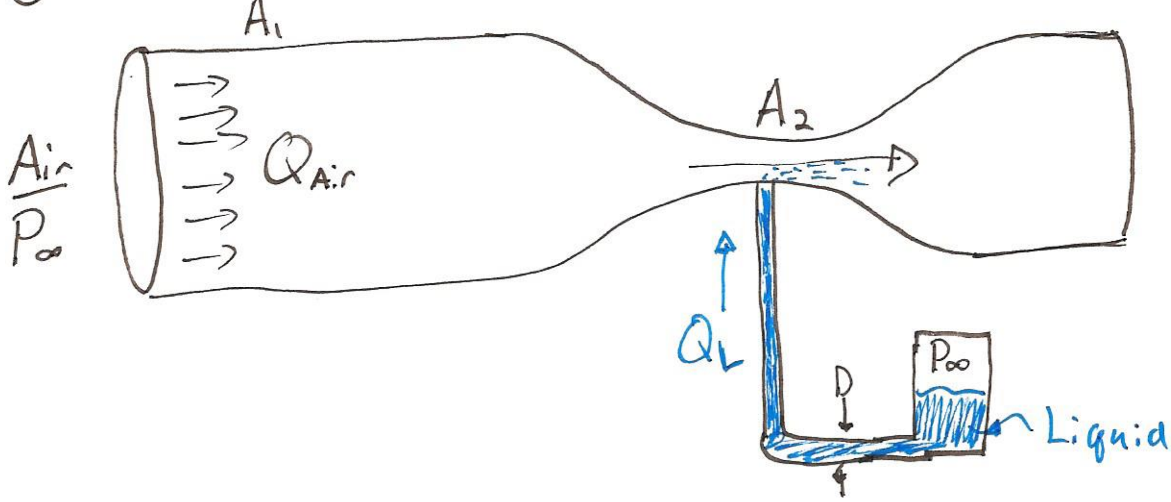
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