# Boundary layers, drag, and internal flows

## I. DRAG ESTIMATES

The following problems are estimates only. Clearly state what you assumed.

1. How much power is lost to drag on your favorite car going 55 mph? 75 mph? If drag were the only loss, what would the top speed be based on the power output of the engine?
2. A reporter is standing outside during a hurricane. How high does the wind speed need to be before the reporter cannot stand normally (straight up with both feet together) ?

## II. BOUNDARY LAYERS

1. A passenger plane is flying at cruise altitude of 30,000 ft. How far from the leading edge of the wing does the boundary layer become turbulent (if at all) ? Considering the flow along the fuselage, what is the boundary layer thickness on the fuselage near the back of the plane? This is an estimate, state your assumptions clearly.
2. The coefficient of friction for a laminar boundary layer is given as $C_f = 0.664/\sqrt{\mathrm{Re}_x}$. For a turbulent boundary layer it is empirically given as $C_f = 0.0592/\mathrm{Re}_x^{1/5}$. The coefficient of friction is defined as $C_f(x) = \tau(x)/\frac{1}{2}\rho V^2$, the normalized local shear stress. Derive an expression for the total drag coefficient of a flat plate(i.e. integrate the local expression for the total- and I won't judge if you use Wolfram alpha). Since the flow transitions around $Re = 10^5$, you will need to properly add the laminar and turbulent solutions together.
3. Using the expression from the previous problem, create a dimensional plot of the total drag force (per unit width) as a function of plate length for flow in air. Plot a family of curves for velocities of 1, 10, and 100 m/s. Use log scales to make everything fit nicely on one graph. This exercise is just to give you a feel for the order of magnitude of flat plate drag force.

## III. PIPE FLOW

1. A 1/2 inch pipe has water flowing through it at 10 liters per minute. The pipe is copper which has a typical roughness of 0.002 mm. What is the pressure drop required to drive this flow in a 10 meter length of pipe?
2. If we double the applied pressure in the previous problem, what is the resulting flow rate?

## IV. TURBULENT INTERNAL FLOWS

In class we will demonstrate the experimental setup. Essentially air blows out of an opening in a flat plate. Another plate is brought close to the opening and this plate is pulled upward and levitates like an upside down air hockey puck, even though the air is blowing outward. The behavior seems counterintuitive, but is easily explained with basic fluid dynamics.

- **Derive the basic equations**. To make progress, we will simplify the Navier Stokes equations significantly. Start with the Navier-Stokes equations for an incompressible, Newtonian fluid in cylindrical coordinates. Assume flow is only radial, $v_r$, such that the theta and z components of the velocity are zero. Also assume the problem is symmetric and all theta derivative drop out. The radial momentum equation and conservation of mass with these simplifications are

$$\frac{\partial(rv_r)}{\partial r} = 0,$$

$$\rho v_r \frac{\partial v_r}{\partial r} = -\frac{\partial P}{\partial r} + \mu \frac{\partial^2 v_r}{\partial z^2}.$$

Conservation of mass therefore tells us that

$$v_r = \frac{Q}{2\pi r h},$$

where $Q$ is the total flow rate and $h$ is the gap height. We can now take this simplified momentum equation and group some terms together to obtain,

$$\frac{\partial(\frac{1}{2}\rho v_r^2)}{\partial r} = -\frac{\partial P}{\partial r} + \mu\frac{\partial^2 v_r}{\partial z^2},$$

or

$$\frac{\partial}{\partial r}\left(\frac{1}{2}\rho v_r^2 + P\right) = \mu\frac{\partial^2 v_r}{\partial z^2}.$$

We can see then that if the viscosity is taken as zero, then we have Bernoulli's equation along a streamline from the center to the outer edge. However, lets leave in viscosity and integrate the equation over the gap height,

$$\int_0^H \left(\frac{\partial}{\partial r}\left(\frac{1}{2}\rho v_r^2 + P\right)\right) dz = 2\mu \left.\frac{\partial v_r}{\partial z}\right|_{z=0}$$

If we define $\bar{v}_r$ as the integrated, averaged velocity then we have

$$\frac{\partial}{\partial r}\left(\frac{1}{2}\rho \bar{v}_r^2 + P\right) = \frac{2\mu}{H}\frac{\partial v_r}{\partial z}$$

Note that $\tau = \mu\frac{\partial v_r}{\partial z}$ is the shear stress at the solid surface,

$$\frac{\partial}{\partial r}\left(\frac{1}{2}\rho \bar{v}_r^2 + P\right) = \frac{2\tau}{H}.$$

If the flow were laminar we could have some hope of solving for the shear stress exactly. However, the flow is certainly turbulent and the shear stress $\tau$ is given by the empirical parameter friction factor, $f$. Empirically, for flow in a pipe we describe the shear stress as $4\tau = \frac{1}{2}\rho v^2 f$, where $L$ is the length and $D$ the diameter. The factor of 4 is just by convention. To simplify things we will assume the friction factor is a constant and will not bother to account for its dependence on Reynolds number.

The resulting equations for conservation of mass and momentum allow us to easily solve for the pressure and radial velocity as a function of radius;

$$v_r = \frac{Q}{2\pi r h}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{2}\rho \bar{v}_r^2 + P\right) = \frac{\rho \bar{v}_r^2 f}{4H}$$

- **Solve the equations**. The equations are simple enough that you can integrate them directly and solve for the pressure as a function of radius. The boundary conditions is that the pressure at the outer radius matches that of the atmosphere. To get the total force on the plate you will need to integrate the pressure over the entire surface of the plate. This force needs to balance the weight of the plate in order for the system to remain stable. You can use Wolfram alpha or some other symbolic manipulator to help keep things straight if that helps, but it is easy enough to do by hand.
- **Compare to experiments.** You should now be able to compare what this relatively simple model predicts versus what the experiment says. We will provide data for two different configurations providing total flow, pressure distribution, and plate height. You should be able to select a single friction factor that models both data sets reasonably well, though you can use a slightly different factor for each experiment.