Week 7: Solutions to Navier-Stokes

I. PROBLEMS

The following three problems are ones that have shown up in my own research over the past few years. While the original problem I was interested was supposedly more complicated than the ones I give here, solving these simpler problems gave me insight that allowed me to reduce something I thought was complicated to something simpler. I'll discuss the applications in class.

1. Normally in fluid dynamics we assume the no-slip condition at a solid wall. This condition is empirically true for most cases, though there are exceptions where slip has been observed. An interesting case where there is "apparent" slip is when you have charged walls, ions in the fluid (like a dissolved salt solution), and applied electric fields. A material like glass when it comes in contact with an aqueous solution will often have surface groups which lose a proton. The solid wall then becomes negatively charged.

   This negatively charged wall will attract positive ions from the aqueous salt solution to screen the surface charge. However, thermal motion at the molecular scale will keep the ions from forming a perfect layer and a diffuse layer of positive charge will form in the liquid. This layer will have a thickness on the order of 10 nm (that's nanometers) for a 1 mM (milli-molar) salt solution, the length scale is called the Debye length. More than 10 nm away from the wall, the solution will be neutral with the number of negative ions balanced by the number of positive.

   In this case, if you apply an axial electric field down the channel, the electric field will exert a force on the positively charged ions near the wall and the resulting force will drag the fluid along. This electrical force will only be exerted in this charged layer which is 10 nm thick. Thus, even though the fluid at the wall will obey the no slip condition, the fluid 10nm away will be in motion. The net result after is that the flow in a channel which is much larger than 10 nm will have an apparent slip velocity at the wall. This slip velocity is proportional to the electric field. The formula for the slip is $U_{slip} = -\epsilon_0 \epsilon_w \zeta E/\mu$, where ϵ_0 is the vacuum permittivity, ϵ_w is the relative permittivity for water (approximately 80), E is the applied electric field, μ is the dynamic viscosity, and ζ is the so-called zeta-potential, the voltage difference between the wall and the solution far away. In typical glass/water systems $\zeta \approx -100$ mV. This theoretical result is over 100 years old, it is called the Helmholtz-Smoluchowski equation. The effect, called electroosmosis has been known experimentally for over 200 years and was first observed in porous clays. Sometimes it is more convenient to measure the combination of parameters and call it the electroosmotic mobility, $b = -\epsilon_0 \epsilon_w \zeta/\mu$, and simply express the slip velocity as $U_{slip} = bE$.

   If we have a long, thin tube with no applied pressure difference and the electroosmotic (EO) slip velocity is a constant, you would find trivially, that the solution to the Navier-Stokes equation is a constant fluid velocity. All the fluid moves down the tube at the slip velocity. In recent years, these electroosmotic flows have been exploited in a number of modern micro and nanoscale systems. Given the results you have already found for pressure driven flow, you saw that the hydraulic resistance scales poorly as the channel or pipe becomes very small. Thus in small systems, electroosmosis is a practical and efficient pump. It is also a simple pump because it has no moving parts, just apply a voltage across your channel.

   While all this background may sound complicated, the final result is quite simple. From the system parameters we can compute the slip velocity. We then solve the flow problem using the slip velocity as the boundary condition for the Navier Stokes. At a solid surface, the normal component of the velocity is zero while the tangential velocity is given as the slip value.

   OK. That was all background. Now the actual problems.

   - Consider a small tube of radius r and length L. Assume a pressure ΔP and voltage ΔV difference are applied across the tube. The tube is long such that we can consider the flow invariant in the axial direction. The pressure gradient $dP/dx = \Delta P/L$ and the electric field $E = \Delta V/L$ are constants along the length of the channel. We will assume the electroosmotic mobility is a known constant (usually comes from measurement). Thus, the apparent slip velocity at the wall has a known value U_{slip}. Solve the Navier Stokes equations for this case. State the velocity profile as a function of r and how it depends on the system parameters, $r, L, \Delta P, \Delta V, \mu$, and U_{slip}. Compute the average fluid velocity i.e. $U_{ave}\pi r^2 = Q$ where Q is the total volumetric flow rate. Write your expression for the average velocity in the form, $U_{ave} = A\Delta P + B\Delta V$, where A and B are constants that you will figure out. If you are really smart you can use your result from Problem 2 and realize that superposition works in this case!

- Plot some sample velocity profiles as $u(r)/U_{slip}$ (i.e. velocity as a function of radius). See if you can think of a good way to plot the data such that there is a single parameter that captures the relative strength of the electrically and pressure driven flows.
- Consider a tube of radius 10 microns and length 10 mm, containing an aqueous solution. The measured mobility is $b = 7 \times 10^{-8} \frac{1}{\mathrm{V \cdot s}}$ (that's 1/(Volts-Seconds) and the electric field is in Volts/m) The applied voltage is 5 kV. Compute the mean flow velocity if there is no applied pressure gradient. Compute the applied pressure needed to obtain the same mean flow velocity if there were no applied electric field. Repeat the above calculations if the tube has a radius of 1 micron. Comment on whether the size of the pressure gradient is "big" or not.
- In some applications, we can intentionally coat or modify the solid surface to change the surface charge and thus change the apparent slip velocity. Consider a case where the channel from $0 < x < L_1$ has slip velocity $U_{s,1}$ and $L_1 < x < L$ has slip velocity $U_{s,2}$. Consider flow in such a system where we have two regions of two different slip velocities, but NO applied pressure. Since the channel is long compared to the other scales, we can assume everything is uniform in the axial direction and simply assume we have two channels which are connected together in series.

  Since the slip velocity is different in the two regions, but the total flow through any cross section of the channel must be the same, something has to give. What happens is that while the net applied pressure is zero, each region will have to acquire an internally generated pressure gradient. This pressure gradient will enhance or retard the electrically driven slip flow such that each region has the same total flow rate. The pressure gradient in each region is a constant and described as $dP/dx|_1 = \Delta P_1/L_1$, and $dP/dx|_2 = \Delta P_2/L_2$ where the total applied pressure is zero, $\Delta P_1 + \Delta P_2 = 0$. The electric field is a constant in both regions, $E = E_1 = E_2 = \Delta V/L$, thus the slip velocity is constant along the length of the channel. The average flow in both regions must be equal, $U_{ave,1} = U_{ave,2}$. Use all the above relationships as well as your formula from part 1, $U_{ave} = A\Delta P + B\Delta V$, to determine the pressure gradient in each region.
- Assume the following parameters. A tube of radius 10 microns and length 10 mm. The electroosmotic mobility is measured to be $b = 7 \times 10^{-8} \frac{1}{\mathrm{V \cdot s}}$ in region 1. In region 2 it is reduced to $b = 0.7 \times 10^{-8} \frac{1}{\mathrm{V \cdot s}}$. The applied voltage is 5000 Volts. Compute and plot the velocity profile in both regions as a function of r in units of mm/s. Plot the velocity profile for $L_1/L_2 = 1/4$, 1, and 4. Compute the average fluid velocity in both regions for all cases of L_1/L_2. Note that the average velocity in region 1 and region 2 better be equal in each case. Check your result and make sure that the limiting cases work, i.e. L_1 and L_2 going to zero limit to the result you derived in part 1.

2. In industrial applications of pumping viscous fluids, it is often useful to add a small amount of immiscible, less viscous fluid to reduce pumping losses. In pipe flow, sometimes it is possible to achieve a state where the more viscous fluid will assume the core of the pipe and the less viscous fluid will encapsulate it and remain in contact with the wall (this is the thermodynamically favored state). The less viscous fluid acts to lubricate the flow. This problem is discussed in the book.

   We will consider a variation on this problem. Consider 2D channel of height H. From $0 < y < h$, we have fluid 1 with a high viscosity, μ_1. From $h < y < H$ we have water with a viscosity μ_2. The flow is pressure driven with a constant pressure drop of $\Delta P/L$.

   - Write the NS equations and cross out terms that are zero. Write the final simplified Navier stokes equations. We will assume that we are looking at the steady state behavior in a very long channel.
   - Formulate the problem - i.e. state the final equation and boundary conditions. Treat the problem as two separate problems, one for each fluid. Match the boundary conditions at $y = h$. The velocity and stress must be equal at the interface between the two fluids.
   - Solve the problem by integrating with respect to y and apply the boundary conditions.
   - Calculate the flow rate of each fluid.
   - Calculate the overall hydraulic resistance, $R = \Delta P/Q$. Plot the effective resistance as a function of $Q_2/(Q_1 + Q_2)$. Normalize your resistance plot to the resistance if the flow were all water. Plot curves for different values of the viscosity ratio between the two fluids.

3. A 2D block is sliding at constant speed U. The block has length $2L$ and is parallel to a stationary wall below it. Fluid fills the small gap between the block and the stationary wall. The block has a step in it such that the first half of the block from $0 > x < L$ has a gap of height h and the second half of the block has a gap of height $h/2$. The length L is large compared to the gap height so we are safe ignoring effects that may occur

right at the entrance and at the step. In the long regions of constant height, at first glance it would seem logical to conclude that the flow should be the Couette solution and have a linear velocity profile. However, if there were a linear velocity profile in the two sections, then the total mass flow through the two sections would differ - thus this is not a possible solution. Therefore a pressure gradient must develop in the gap to drive a parabolic pressure driven flow profile superimposed on the linear one profile. This internal pressure gradient develops even though there is no applied pressure across the length of the gap. In one section under the block, the flow rate is increased by the internal pressure gradient and in the other it must be suppressed. This situation is analogous to the first problem in this homework.

- Before trying to solve the problem, sketch the velocity fields and pressure as a function of length for this problem. Try to reason what the answer is first.
- For flow in a 2D slot, derive the general velocity profile that results from a gap of uniform height with an applied pressure gradient and a sliding wall.
- Integrate this expression across the gap to obtain the total flow rate. Express the flow rate as two terms, one due to pressure and one due to the moving wall.
- For the block sliding problem with two heights, equate your expression for the total flow in the two regions in order to solve for the pressure distribution under the block. Write out the expression for the maximum pressure.
- Confirm your results using Comsol at a low Reynolds number (i.e. use Normalized units and set the density and viscosity to 1). Work in dimensionless terms. Set the gap heights to one and one-half, and the total length to 20.
- Imagine we now have 4 constant height sections, which are at heights h, $\frac{7}{8}$ h, $\frac{3}{4}$ h, and $\frac{1}{2}$ h. Each section is $L/2$ in length. See if you can guess the solution. Try it in Comsol to confirm/deny your guess.
- Compare the previous result in Comsol to that where the height varies linearly from h to $h/2$ over the length 2L.