Vector calculus

All problems due Monday, Sept 22, 2014.

I. READING

This week you should read

- Chapter on vector calculus

II. VECTOR CALCULUS - 10 EACH

1. Consider the scalar field,

$$f = \exp(-10(x^2 + y^2))$$

   - Plot the contours of the scalar field on the domain $-1 < x < 1$, $-1 < y < 1$.
   - Compute the gradient and plot the vector field ∇f.
   - Compute the Laplacian ∇^2 and plot contours of the result.

2. Consider the vector function given by,

$$\mathbf{f} = x\mathbf{i} + y\mathbf{j}$$

   - Plot the vector field.
   - Compute the divergence.
   - Consider a circle of radius R centered at the origin; compute $\mathbf{f} \cdot \mathbf{n}$.
   - Evaluate $\int \mathbf{f} \cdot \mathbf{n} dS$ along the same circle.
   - Evaluate $\int \nabla \cdot \mathbf{f} dV$ for the same circle.
   - Consider a square of side length L centered at the origin, $\int \mathbf{f} \cdot \mathbf{n} dS$ along the boundary of the square.
   - Convert the vector field to one described in cylindrical coordinates.
   - Look up the divergence operator in cylindrical coordinates and recompute the divergence of the field.

3. Consider the vector function given by,

$$\mathbf{f} = \frac{-y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2 + y^2}}$$

   - Plot the vector field.
   - Compute the divergence.
   - Consider a circle of radius R centered at the origin; compute $\mathbf{f} \cdot \mathbf{n}$.
   - Convert the vector field to one described in polar coordinates.
   - Look up the divergence operator in polar coordinates and recompute the divergence of the field.

4. Consider the vector function given by,

$$\mathbf{f} = -x\mathbf{i} + y\mathbf{j}$$

   - Plot the vector field.
   - Compute the divergence.
   - Consider a square of of side L centered at the origin; compute $\mathbf{f} \cdot \mathbf{n}$.
   - Evaluate $\int \mathbf{f} \cdot \mathbf{n} dS$ along the same square.

5. Consider the scalar field,

$$\phi = x + \frac{x}{x^2 + y^2}$$

This field represents the ideal velocity potential for flow around a cylinder in 2D. The velocity vectors are defined as $\nabla\phi = \mathbf{v}$. Investigate the field on the domain $-5 < x < 5$, $-5 < y < 5$ and where $x^2 + y^2 > 1$

- Plot the contours of the scalar field.
- Compute the gradient and plot the vector field $\nabla\phi$.
- Compute the Laplacian ∇^2 and plot contours of the result.

6. Demonstrate that $\int \nabla \cdot \mathbf{f} dV = \int \mathbf{f} \cdot \mathbf{n} dS$. You can just do this in 2D. Draw a square and write out all the terms for computing $\int \mathbf{f} \cdot \mathbf{n} dS$ on the four sides. Use the fundamental theorem of calculus $\int_a^b \frac{\partial f}{\partial x} dx = f(b) - f(a)$ to convert the surface integral to the volume one.

7. Use the divergence theorem to derive conservation of mass for a fluid flow. Namely;

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v})$$

To do this you will need to start with the fact that the volume integral of the fluid's mass density ρ is the total mass inside the volume. The total mass contained in this arbitrary volume can only change is there is a net flux of matter coming in and out of the volume. The net flux which mass crosses a point on the surface in the flow is $\rho\mathbf{v} \cdot \mathbf{n}$ where $\mathbf{n}$ is the surface's normal vector and $\mathbf{v}$ is the velocity vector. Use these relations to develop a relationship between a volume and surface integral. Apply the divergence theorem to obtain the result above.

8. Demonstrate the

- $\nabla \times (\nabla\phi) = 0$ where ϕ is a scalar field.
- $\nabla \cdot (\nabla \times \mathbf{v}) = 0$ where $\mathbf{v}$ is a vector field.

9. The operator $\mathbf{v} \cdot \nabla$ where $\mathbf{v}$ is the vector field of the fluid velocity will show up a lot in our equations in this course. Expand the operator out in component form where the components of velocity are $\mathbf{v} = [u, v, w]$ for the x, y, and z directions.